Singularity Analysis of Curvature Flow of Curves on a Riemannian Surface

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Abstract: In this paper, we consider evolution of embedded curves by curvature flow in a compact Riemannian surface. Let γ be a closed embedded curve evolving under the curvature flow in a compact surface M. If a singularity develops in finite time, then the curve shrinks to a point. Therefore, when t is close enough to the blow- up time, we may assume that the curve is contained in a small neighborhood of the collapsing point on the surface. Using a local conformal diffeomorphism $\phi: U(\subseteq M) \to U' \subseteq \mathbb{R}^2$ between compact neighborhoods, we get a corresponding flow in the plane which satisfies the following equation: $\frac{\partial \gamma'}{\partial t} = (\frac{k'}{J^2} - \frac{\nabla_N J}{J^2})N'$

where $\gamma'(p,t) = \phi\big(\gamma(p,t)\big), k'$ is the curvature of γ' in U', N' is the unit normal vector, and the conformal factor J is smooth, bounded and bounded away from 0. We define the extrinsic and intrinsic distance functions $d, l : \Gamma \times \Gamma \times [0,T] \to \mathbb{R}$ by

$d(p,q,t) \coloneqq |\gamma(p,t) - \gamma(q,t)|_{\mathbb{R}^2}$ and $l(p,q,t) \coloneqq \int_p^q ds_t = s_t(q) - s_t(p)$

where Γ is either S^1or an interval. We also define the smooth function $\psi: \mathrm{S}^1 \times \mathrm{S}^1 \times [0,\mathrm{T}] \to \mathbb{R}$ by

$$\psi(p,q,t) \coloneqq \frac{L(t)}{\pi} \sin\left(\frac{l(p,q,t)\pi}{L(t)}\right).$$

We use the distance comparison $\frac{d}{l}$ and $\frac{d}{\psi}$ to prove the following theorem.

Main Theorem: *Let γ be a closed embedded curve evolving by curvature flow on a smooth compact Riemannian surface. If a singularity develops in finite time, then the curve converges to a round point in the C^∞ sense.*

This extends Huisken's distance comparison technique for curvature flow of embedded curves in the plane. Hamilton used isoperimetic estimates techniques to prove that when a closed embedded curve in the plane evolves by curvature flow the curve converges to a round point and Zhu used Hamilton's isoperimetric estimates techniques to study asymptotic behavior of anisotropic curves flows.