COHERENT STATES FOR AN ABSTRACT HAMILTONIAN WITH A GENERAL SPECTRUM

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Abstract: Temporally stable coherent states are discussed for an abstract Hamiltonian with a general spectrum. Statistical quantities related to the coherent states are calculated. As special cases of the construction, coherent states for some well-known Hamiltonians, namely; Harmonic oscillator, Isotonic oscillator, pseudoharmonic oscillator, Infinite well potential, Pöschl-Teller potential, Eckart potential are indicated. Quaternion version of temporally stable coherent states is also worked out.

Keywords: coherent states, Hamiltonian.

1. Introduction

Following the method proposed by Gazeau and Klauder [6] to construct temporally stable coherent states, CS for short, in recent years, several classes of CS were constructed for quantum Hamiltonians [2],[5],[12]. The spectrum $E(n)$ of several solvable quantum Hamiltonians is a polynomial of the label n. In this letter, we discuss CS with a general polynomial $E(n)$,

$$E(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0,$$

of degree k, which is considered as the spectrum of an abstract Hamiltonian. As special cases of our construction we obtain CS for the quantum Hamiltonians indicated in the abstract.

2. Gazeau-Klauder coherent states

Let us introduce the general features of Gazeau-Klauder CS. Let H be a Hamiltonian with a bounded below discrete spectrum $\{e_n\}_{n=0}^{\infty}$ and it has been adjusted so that $H \geq 0$. Further assume that the eigenvalues e_n are non-degenerate and arranged in increasing order, $e_0 < e_1 < ...$ For such a Hamiltonian, the so-called *Gazeau-Klauder coherent states* (GKCS for short) are defined as

$$| J,\alpha\rangle = \mathcal{N}(J)^{-1} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{\kappa(n)}} e^{-ie_n\alpha} \eta_n \tag{2.1}$$

where $J \geq 0$, $-\infty \leq \alpha \leq \infty$, $\{\eta_n\}_{n=0}^{\infty}$ is the set of eigenfunctions of the Hamiltonian and $\kappa(n) = e_1 e_2 \dots e_n = e_n!$. In order to be GKCS the states (2.1) need to satisfy the following:

(a) For each J,α the state is normalized, i.e., $\langle J,\alpha \mid J,\alpha\rangle = 1$;

(b) The set of states $\{| J,\alpha\rangle : J \in [0,\infty), \alpha \in (-\infty,\infty)\}$ satisfies a resolution of the identity

$$\int_0^{\infty}\int_{-\infty}^{\infty} | J,\alpha\rangle\langle J,\alpha \mid d\mu(J,\alpha) = I$$

where $d\mu(J,\alpha)$ is an appropriate measure.

(c) The states are temporally stable, i.e., $e^{-iHt} \mid J, \alpha\rangle = \mid J, \alpha + t\rangle$;
(d) The states satisfy the action identity, i.e., $\langle J, \alpha \mid H \mid J, \alpha\rangle = J$.

The condition (d) requires $e_0 = 0$. In the case where only the conditions (a)-(c) are satisfied the resulting CS may be phrased as "temporally stable CS".

2.1. **Abstract approach.** In this section, we manipulate GKCS of type (2.1) in a somewhat abstract way. For this recall the basic definition of the canonical CS [1]:

$$\mid z\rangle = e^{-r^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \mid n\rangle \tag{2.2}$$

where $z \in \mathbb{C}$, the complex plane and $\{\mid n\rangle\}_{n=0}^{\infty}$ is the Fock space basis. As a generalization of (2.2) the so-called non-linear CS are defined [10] by

$$\mid z\rangle = \mathcal{N}(|z|)^{-1} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{x_n!}} \xi_n \tag{2.3}$$

where $z \in D$, an open subset of $\mathbb{C}$, $\mathcal{N}(|z|)$ is the normalization factor, $\{\xi_n\}_{n=0}^{\infty}$ is an orthonormal basis of an abstract separable Hilbert space $\mathfrak{H}$, $x_1, x_2, ...$ a sequence of positive real numbers, $x_n! = x_1 ... x_n$, the generalized factorial and, by convention, $x_0 = 0$, $x_0! = 0! = 1$ (notice that in (2.3) it is custom to take $\mathcal{N}(|z|)^{-1/2}$ and $x_0! = 1$, but to be consistent with [6] we take $\mathcal{N}(|z|)^{-1}$ and $x_0 = 0$). If $x_n!$ is given by $\rho(n)$, a positive real number x_n can be obtained as follows:

$$x_n = \frac{\rho(n)}{\rho(n-1)}, \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{2.4}$$

Then

$$\rho(n) = x_n x_{n-1} \ldots x_1 = x_n!, \tag{2.5}$$

and $\rho(0) = x_0! := 0! = 1$. The generalized annihilation, creation and number operators defined on the Hilbert space $\mathfrak{H}$ with respect to the basis $\{\xi_n\}$ can be given by (see [1])

$$\begin{aligned} \mathfrak{a}\xi_n &= \sqrt{x_n}\xi_{n-1}, \quad \text{with} \quad \mathfrak{a}\xi_0 = 0, \\ \mathfrak{a}^\dagger\xi_n &= \sqrt{x_{n+1}}\xi_{n+1}, \\ \mathfrak{n}\xi_n &= x_n\xi_n, \quad (\mathfrak{n} = \mathfrak{a}^\dagger\mathfrak{a}) \end{aligned} \tag{2.6}$$

and the commutators take the form

$$\begin{aligned} \left[\mathfrak{a}, \mathfrak{a}^\dagger\right]\xi_n &= (x_{n+1} - x_n)\xi_n, \\ \left[\mathfrak{n}, \mathfrak{a}^\dagger\right]\xi_n &= (x_{n+1} - x_n)\mathfrak{a}^\dagger\xi_n, \\ \left[\mathfrak{n}, \mathfrak{a}\right]\xi_n &= (x_{n-1} - x_n)\mathfrak{a}\xi_n. \end{aligned} \tag{2.7}$$

The annihilation operator satisfies the usual relation $\mathfrak{a} \mid z\rangle = z \mid z\rangle$. Under the commutator bracket, these three operators generate a Lie algebra which is the so-called generalized oscillator algebra. Since $\mathfrak{n} = \mathfrak{a}^\dagger\mathfrak{a}$ and

$$\mathfrak{n}\xi_n = x_n\xi_n,$$

we can consider $\mathfrak{n}$ as a Hamiltonian, $\{x_n\}_{n=0}^{\infty}$ as its non-degenerate spectrum and $\{\xi_n\}_{n=0}^{\infty}$ as its eigenfunctions. Further, if $x_0 < x_1 < x_2 < ...$ then in analogy to the GK construction we can have GKCS. That is,

$$\mid J, \alpha\rangle = \mathcal{N}(J)^{-1} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{x_n!}} e^{-ix_n\alpha}\xi_n \tag{2.8}$$

It is straightforward to verify that the states in (2.8) are temporally stable under the action of the time evolution operator

$$U(t) = e^{-i\mathfrak{n}t}$$

and since $x_0 = 0$ we have the action identity,

$$\langle J, \alpha \mid \mathfrak{n} \mid J, \alpha \rangle = J.$$

Thus the states (2.8) form a class of GKCS if the states are normalized, i.e.,

$$\langle J, \alpha \mid J, \alpha \rangle = 1,$$

which is guaranteed if

$$\mathcal{N}(J)^2 = \sum_{n=0}^{\infty} \frac{J^n}{x_n!} < \infty, \tag{2.9}$$

where the radius of convergence of the series is $R = \lim_{n\to\infty} \sqrt[n]{x_n}$, and provide a resolution of the identity, i.e.,

$$\int_0^R \int \mid J, \alpha\rangle\langle J, \alpha \mid \Xi(J) dJ d\alpha = I, \tag{2.10}$$

where $\Xi(J) = \mathcal{N}(J)^2\lambda(J)$ is a density function and $\lambda(J)$ is an auxiliary density. Further, the integral on α is defined by

$$\int ...d\alpha = \lim_{\delta\to\infty} \frac{1}{2\delta} \int_{-\delta}^{\delta} ...d\alpha.$$

Notice that,

$$\int e^{-i\alpha(x_n - x_l)} d\alpha = \lim_{\delta\to\infty} \frac{1}{2\delta} \int_{-\delta}^{\delta} e^{-i\alpha(x_n - x_l)} d\alpha = \begin{cases} 0 & \text{if} \quad x_n \neq x_l \\ 1 & \text{if} \quad x_n = x_l \end{cases}$$

By a straightforward calculation one can see that the identity (2.10) is satisfied if one has,

$$\int_0^R J^n \lambda(J) dJ = x_n! \tag{2.11}$$

Further it may be interesting to notice that the algebra generated by the operators $\{\mathfrak{a}, \mathfrak{a}^\dagger, \mathfrak{n}\}$ and its deformations (up to isomorphisms) can serve as a dynamical algebra of the Hamiltonian $\mathfrak{n}$.

In the case where one knows the spectrum and the eigenfunctions of a Hamiltonian, the projective representation of the Hamiltonian can be written. For the states (2.8) it can be written as

$$H = \sum_{n=0}^{\infty} x_n \mid \xi_n\rangle\langle\xi_n \mid . \tag{2.12}$$

For this Hamiltonian we have $H\xi_n = x_n\xi_n;\ \forall n \geq 0$.

2.2. **GKCS quaternionic extension.** Here we present quaternionic extension of GKCS as vector coherent states on an abstract separable Hilbert space tensored with $\mathbb{C}^2$. Even though possible physical applications of these CS may be worked out for the systems presented in [15, 3], we shall not touch them in this manuscript. Further, it might be of interest to carry out the following procedure on a separable abstract left or right quaternionic Hilbert space. However, a quaternionic wave function on a quaternionic Hilbert space has not attained a clear meaning in quantum physics yet. Keeping the above points in mind let us proceed with the construction.

Let $\mathbf{q}$ be a quaternion and $\mathbf{p}$ be a 2×2 Hermitian matrix. We intend to have temporally stable VCS as follows.

$$| \mathbf{q}, \alpha\mathbf{p}, j\rangle = \mathcal{N}(\mathbf{q})^{-1}\sum_{m=0}^{\infty}\frac{\mathbf{q}^{m/2}}{\sqrt{y_m!}}e^{-iy_m\alpha\mathbf{p}}\chi^j\otimes\phi_m\in\mathbb{C}^2\otimes\mathfrak{H},\quad j=1,2. \tag{2.13}$$

where χ^1,χ^2 is the natural basis of $\mathbb{C}^2$, $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal basis for the abstract separable Hilbert space $\mathfrak{H}$, and $\{y_m\}$ is a positive sequence of real numbers with $y_0 < y_1 < y_2 ....$ Further a remark is in order: A quaternion has many square roots, in order to be unique with the definition of GKCS we need to work with a fixed square root.

2.2.1. *Normalization.* As in the case of VCS of [15] we normalize the states as

$$\sum_{j=1}^{2}\langle \mathbf{q},\alpha\mathbf{p},j \mid \mathbf{q},\alpha\mathbf{p},j\rangle = 1.$$

which requires

$$\begin{aligned}\sum_{j=1}^{2}\langle \mathbf{q},\alpha\mathbf{p},j \mid \mathbf{q},\alpha\mathbf{p},j\rangle &= \mathcal{N}(\mathbf{q})^{-2}\sum_{j=1}^{2}\sum_{m=0}^{\infty}\frac{\langle \mathbf{q}^{m/2}e^{-iy_m\alpha\mathbf{p}}\chi^j \mid \mathbf{q}^{m/2}e^{-iy_m\alpha\mathbf{p}}\chi^j\rangle}{y_m!}\\ &= \mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{\mathrm{Tr}[(\mathbf{q}^{m/2}e^{-iy_m\alpha\mathbf{p}})(\mathbf{q}^{m/2}e^{-iy_m\alpha\mathbf{p}})^{\dagger}]}{y_m!}\\ &= \mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{\mathrm{Tr}[\mathbf{q}^{m/2}(\mathbf{q}^{m/2})^{\dagger}]}{y_m!}\\ &= 2\mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{|\mathbf{q}|^m}{y_m!}\end{aligned}$$

that is

$$\mathcal{N}(\mathbf{q})^2 = 2\sum_{m=0}^{\infty}\frac{|\mathbf{q}|^m}{y_m!}. \tag{2.14}$$

2.2.2. *Resolution of the identity.* Let $\mathcal{D}(\mathbf{p})$ be the domain of variables of $\mathbf{p}$ and $d\mathbf{p}$ be the probability measure on it. Observe that

$$\int_{\mathcal{D}(\mathbf{p})}\int e^{-i(y_m-y_l)\alpha\mathbf{p}}d\alpha d\mathbf{p} = \begin{cases}1 & \text{if} \quad m=l\\ 0 & \text{if} \quad m\neq l\end{cases}$$

Let us make the following identification

$$|\cdot| : \mathbb{H}\longrightarrow\mathbb{R}^+ \quad \text{by} \quad \mathbf{q}\mapsto|\mathbf{q}|=t,$$

where $\mathbb{H}$ is the quaternion algebra. For a resolution of identity condition, let

$$d\mu(t,\mathbf{p},\alpha) = \mathcal{N}(|\mathbf{q}|)^2\lambda(t)dtd\mathbf{p}d\alpha.$$

Then we have

$$\sum_{j=1}^{2}\int_0^\infty\int_{\mathcal{D}(\mathbf{p})}\int \mid \mathbf{q},\alpha\mathbf{p},j\rangle\langle\mathbf{q},\alpha\mathbf{p},j\mid d\mu(t,\mathbf{p},\alpha)$$

$$= \sum_{j=1}^{2}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty\int_{\mathcal{D}(\mathbf{p})}\int \frac{1}{\sqrt{y_m!y_l!}} \mid \mathbf{q}^{m/2}e^{-i\alpha y_m\mathbf{p}}\chi^j\rangle\langle\mathbf{q}^{l/2}e^{-i\alpha y_l\mathbf{p}}\chi^j \mid$$
$$\otimes \mid \phi_m\rangle\langle\phi_l \mid \lambda(t)dtd\mathbf{p}d\alpha$$

$$= \sum_{j=1}^{2}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty\int_{\mathcal{D}(\mathbf{p})}\int \frac{1}{\sqrt{y_m!y_l!}}\mathbf{q}^{m/2}e^{-i\alpha y_m\mathbf{p}} \mid \chi^j\rangle\langle\chi^j \mid (\mathbf{q}^{l/2}e^{-i\alpha y_l\mathbf{p}})^\dagger$$
$$\otimes \mid \phi_m\rangle\langle\phi_l \mid \lambda(t)dtd\mathbf{p}d\alpha$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty\int_{\mathcal{D}(\mathbf{p})}\int \frac{1}{\sqrt{y_m!y_l!}}\mathbf{q}^{m/2}e^{-i\alpha y_m\mathbf{p}}(\mathbf{q}^{l/2}e^{-i\alpha y_l\mathbf{p}})^\dagger\otimes \mid \phi_m\rangle\langle\phi_l \mid \lambda(t)dtd\mathbf{p}d\alpha$$

$$= \sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\int_0^\infty\int_{\mathcal{D}(\mathbf{p})}\int \frac{1}{\sqrt{y_m!y_l!}}\mathbf{q}^{m/2}e^{-i\alpha(y_m-y_l)\mathbf{p}}(\mathbf{q}^{l/2})^\dagger\otimes \mid \phi_m\rangle\langle\phi_l \mid \lambda(t)dtd\mathbf{p}d\alpha$$

$$= \sum_{m=0}^{\infty}\int_0^\infty \frac{|\mathbf{q}|^m}{y_m!}\mathbb{I}_n\otimes \mid \phi_m\rangle\langle\phi_m \mid \lambda(t)dt$$

$$= \sum_{m=0}^{\infty}\int_0^\infty \frac{t^m}{y_m!}\mathbb{I}_n\otimes \mid \phi_m\rangle\langle\phi_m \mid \lambda(t)dt = \mathbb{I}_n\otimes I$$

provided that

$$\int_0^\infty t^m\lambda(t)dt = y_m! \qquad (2.15)$$

2.2.3. *Temporal stability.* As in [15], we define the operators

$$\mathfrak{A} = \mathbb{I}_2\otimes\mathfrak{a}, \quad \mathfrak{A}^\dagger = \mathbb{I}_2\otimes\mathfrak{a}^\dagger, \quad \mathfrak{N} = \mathbb{I}_2\otimes\mathfrak{n}.$$

Since

$$\mathfrak{N}\chi^j\otimes\phi_m = y_m\chi^j\otimes\phi_m$$

$\chi^j\otimes\phi_m$ can be considered as an eigenfunction of $\mathfrak{N}$ with the spectrum y_m. In other words, $\mathfrak{N}$ can be considered as a matrix Hamiltonian. Then

$$U(\tau) = e^{-i\tau\mathfrak{N}}$$

is the time evolution operator. Since

$$U(\tau)\chi^j\otimes\phi_m = e^{-iy_m\tau\mathbb{I}_2}\chi^j\otimes\phi_m$$

we have

$$U(\tau)\mid \mathbf{q},\alpha\mathbf{p},j\rangle = \mid \mathbf{q},\alpha\mathbf{p}+\tau\mathbb{I}_2,j\rangle$$

Thus the states $\mid \mathbf{q},\alpha\mathbf{p},j\rangle$ are temporally stable or this could be an analogue of the temporal stability.

2.2.4. *Action identity.* Let us see an analogue of the action identity. For the quaternionic GKCS a meaningful way of defining the action identity may be as follows:

$$\sum_{j=1}^{2}\langle \mathbf{q},\alpha\mathbf{p},j \mid \mathfrak{N} \mid \mathbf{q},\alpha\mathbf{p},j\rangle = |\mathbf{q}|,$$

which can be verified in the following way.

$$\begin{aligned}
&\sum_{j=1}^{2}\langle \mathbf{q},\alpha\mathbf{p},j \mid \mathfrak{N} \mid \mathbf{q},\alpha\mathbf{p},j\rangle\\
&= \mathcal{N}(\mathbf{q})^{-2}\sum_{j=1}^{2}\sum_{m=1}^{\infty}\frac{1}{y_{m-1}!}\langle \mathbf{q}^{m/2}e^{-i\alpha y_m\mathbf{p}}\chi^j \mid \mathbf{q}^{m/2}e^{-i\alpha y_m\mathbf{p}}\chi^j\rangle\\
&= \mathcal{N}(\mathbf{q})^{-2}\sum_{j=1}^{2}\sum_{m=0}^{\infty}\frac{1}{y_{m}!}\langle \mathbf{q}^{(m+1)/2}e^{-i\alpha y_{m+1}\mathbf{p}}\chi^j \mid \mathbf{q}^{(m+1)/2}e^{-i\alpha y_{m+1}\mathbf{p}}\chi^j\rangle\\
&= \mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{1}{y_{m}!}\mathrm{Tr}[(\mathbf{q}^{(m+1)/2}e^{-i\alpha y_{m+1}\mathbf{p}})(\mathbf{q}^{(m+1)/2}e^{-i\alpha y_{m+1}\mathbf{p}})^{\dagger}]\\
&= \mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{1}{y_{m}!}\mathrm{Tr}[\mathbf{q}^{(m+1)/2}(\mathbf{q}^{(m+1)/2})^{\dagger}]\\
&= 2\mathcal{N}(\mathbf{q})^{-2}\sum_{m=0}^{\infty}\frac{|\mathbf{q}|^{m+1}}{y_{m}!} = |\mathbf{q}|
\end{aligned}$$

2.2.5. *Dynamical algebra.* If $\mathbf{qp} = \mathbf{pq}$ and $y_{m+1} = c + y_m$, for a constant c, then we can have

$$\mathfrak{A} \mid \mathbf{q},\alpha\mathbf{p},j\rangle = \mathbf{q}e^{c\alpha\mathbf{p}} \mid \mathbf{q},\alpha\mathbf{p},j\rangle.$$

Further the algebra generated by $\{\mathfrak{A},\mathfrak{A}^{\dagger},\mathfrak{N}\}$ can be considered as a dynamical algebra of the system governed by the Hamiltonian $\mathfrak{N}$.

3. GKCS for the spectrum E(n)

In this section we discuss GKCS for a Hamiltonian, in the sense of Section 2.1, with the spectrum

$$E(n) = a_k n^k + a_{k-1}n^{k-1} + \cdots + a_1 n + a_0. \tag{3.1}$$

Also note that from (2.14) and (2.15) the following procedure normalizes the quaternionic GKCS and also give a resolution of the identity with $y_n = E(n)$.
Now as we have mentioned earlier, to have the action identity we need to have $E(0) = 0$. In the case where this requirement is violated we need to adjust the spectrum as follows $e_n = E(n) - E(0)$. In this case we get

$$e_n = n(a_k n^{k-1} + ... + a_1).$$

Let $b_1, ..., b_{k-1}$ be the zeros of the polynomial $a_k n^{k-1} + ... + a_1$ (not necessarily distinct) and assume that the zeros are real numbers. Hereby we write

$$e_n = bn(n - b_1)(n - b_2)...(n - b_{k-1}),$$

where b is some constant, and

$$\rho(n)=\prod_{j=1}^{n} e_j = b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n$$

where $\alpha_j = 1-b_j;\ j=1,...,k-1$ and $(\alpha)_n=\Gamma(n+\alpha)/\Gamma(\alpha)$, the Pochhammer symbol. Let $\{\phi_n\}_{n=0}^{\infty}$ be an orthonormal basis of an abstract separable Hilbert space, $\mathfrak{H}$. Let us consider the Hamiltonian

$$H=\sum_{n=0}^{\infty} e_n \mid \phi_n\rangle\langle\phi_n \mid .$$

Thereby, e_n are the eigenvalues of the Hamiltonian H with the eigenfunctions ϕ_n. In the following we construct GKCS for the Hamiltonian H as vectors in the state Hilbert space $\mathfrak{H}$ of H. Let us define a set of states

$$\mid J,\alpha\rangle = \mathcal{N}(J)^{-1}\sum_{n=0}^{\infty}\frac{J^{n/2}}{\sqrt{e_n!}}e^{-ie_n\alpha}\phi_n\in\mathfrak{H}. \tag{3.2}$$

Since $e_0 = 0$ we can easily observe that the states (3.2) are temporally stable under the time evolution operator $U(t) = e^{-i\omega Ht}$ and the action identity can be seen by a straightforward calculation. The normalization requirement $\langle J,\alpha \mid J,\alpha\rangle = 1$ yields

$$\mathcal{N}(J)^2=\sum_{n=0}^{\infty}\frac{(J/b)^n}{\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n} = {}_0F_{k-1}(-;\alpha_1,...,\alpha_{k-1};J/b). \tag{3.3}$$

Since $\lim_{n\to\infty} e_n = \infty$ the series (3.3) converges for all $J \geq 0$. For $J \in [0,\infty)$ and $-\infty < \alpha < \infty$, from (2.10) and (2.11) we see that a resolution of identity holds if there exists a density $\lambda(J)$ satifying

$$\int_0^{\infty} J^n\lambda(J)dJ = b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n. \tag{3.4}$$

From the Mellin transform (see [9], p. 303, formula (37))

$$\int_0^{\infty} x^{s-1}G^{q+1,0}_{p,q+1}\left(x\Big|\begin{array}{ccc} c_1-1, & ..., & c_p-1\\ d_1-1, & ..., & d_q-1,0\end{array}\right)dx=\Gamma(s)\frac{\Gamma(s+d_1-1)...\Gamma(s+d_q-1)}{\Gamma(s+c_1-1)...\Gamma(s+c_p-1)}, \tag{3.5}$$

where

$$G^{q+1,0}_{p,q+1}\left(x\Big|\begin{array}{ccc} c_1-1, & ..., & c_p-1\\ d_1-1, & ..., & d_q-1,0\end{array}\right)$$

is the Meijer-G-function, we conclude that

$$\lambda(J)=\frac{1}{b\prod_{j=1}^{k-1}\Gamma(\alpha_j)}G^{k,0}_{0,k}\left(J/b\Big|\begin{array}{ccc} & - & \\ \alpha_1-1, & ..., & \alpha_{k-1}-1,0\end{array}\right)$$

satisfies (3.4). Thus the states (3.2) form a set of GKCS for the Hamiltonian H.
A dynamical algebra can be defined through the operators of (2.6). In general this algebra is an infinite dimensional Lie algebra.
Quantum revivals are associated with the wave functions. A revival of a wave function occurs when a wave function evolves in time to a state closely reproducing its initial form. Further, the weighting distribution is crucial for understanding the temporal behavior of the wave function [2]. In the case of the states (3.2), the probability of finding the state ϕ_n in the state $\mid J,\alpha\rangle$ is given by

$$P(n,J)=|\langle\phi_n \mid J,\alpha\rangle|^2=\frac{J^n}{b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n {}_0F_{k-1}(-;\alpha_1,...,\alpha_{k-1};J)}.$$

A quantitative estimate is given by the so-called Mandel parameters,

$$Q = \frac{\langle J,\alpha \mid \mathfrak{n}^2 \mid J,\alpha\rangle - \langle J,\alpha \mid \mathfrak{n} \mid J,\alpha\rangle^2 - \langle J,\alpha \mid \mathfrak{n} \mid J,\alpha\rangle}{\langle J,\alpha \mid \mathfrak{n} \mid J,\alpha\rangle}$$

where $\mathfrak{n}\phi_n = e_n\phi_n$. If the photon distribution is Poissonian then $Q = 0$. If $Q < 0$ it is called sub-Poissonian and if $Q > 0$ it is called super-Poissonian [2]. Let us calculate the Mandel parameter for the states (3.2). Since $\mathfrak{n}\phi_0 = 0$ we have

$$\langle J,\alpha \mid \mathfrak{n} \mid J,\alpha\rangle = J$$

and

$$\begin{aligned}\langle J,\alpha \mid \mathfrak{n}^2 \mid J,\alpha\rangle &= \mathcal{N}(J)^{-2}\sum_{n=0}^{\infty}\frac{J^{n+1}e_{n+1}}{e_n!}\\ &= \mathcal{N}(J)^{-2}\sum_{n=0}^{\infty}\frac{J^{n+1}(n+1)(a_k(n+1)^{k-1}+...+a_2(n+1)+a_1)}{b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n}.\end{aligned}$$

Thereby

$$Q = \mathcal{N}(J)^{-2}\sum_{n=0}^{\infty}\frac{J^{n}(n+1)(a_k(n+1)^{k-1}+...+a_2(n+1)+a_1)}{b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n} - J - 1. \tag{3.6}$$

Since, for any finite k,

$$\lim_{n\to\infty}\sqrt[n]{\frac{b^n\Gamma(n+1)(\alpha_1)_n...(\alpha_{k-1})_n}{(n+1)[a_k(n+1)^{k-1}+...+a_2(n+1)+a_1]}} = \infty$$

the series in (3.6) converges for all $J \geq 0$. For a state $\mid \psi\rangle$ of the state Hilbert space the average energy of the system is given by $E = \langle\psi \mid \mathfrak{n} \mid \psi\rangle$. For the states (3.2) the average energy $E = J$.

4. Examples

In the following we will discuss GKCS for some Hamiltonians as special cases of the above construction. Most of these results can be found in the literature.

• *Harmonic oscillator* : The simplest case is the harmonic oscillator Hamiltonian where $e_n = n$ and the state Hilbert space is the Fock space. This case is obtained from (3.1) by taking $k = 1$, $a_1 = 1$, $a_0 = 0$ and assuming that ϕ_n of (3.2) form the Fock space basis. Further $\rho(n) = e_n! = \Gamma(n+1)$, $\mathcal{N}(J)^2 = {}_0F_0(-;-;J) = e^J$,

$$\lambda(J) = G^{1,0}_{0,1}\left(J| \begin{array}{c} - \\ 0 \end{array}\right) = e^{-J}$$

and the Mandel parameter $Q = 0$.

• *Isotonic oscillator* : The spectrum of the isotonic oscillator Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + \frac{A}{x^2} \quad (A \geq 0)$$

is $E_n = 2(2n+\gamma)$ where $\gamma = 1 + \frac{1}{2}\sqrt{1+4A}$, thus $e_n = E_n - E_0 = 4n$. The eigenfunctions of H form an orthonormal basis of the Hilbert space $L^2([0,\infty))$ [7, 13]. We get the spectrum by substituting $k = 1$, $a_1 = 4$, $a_0 = 2\gamma$ in (3.1). In (3.2) we need to take $\phi_n =$ eigenfunction of H. In this case $\rho(n) = 4^n\Gamma(n+1)$, $\mathcal{N}(J)^2 = {}_0F_0(-;-;J/4) = e^{J/4}$ and

$$\lambda(J) = \frac{1}{4}G^{1,0}_{0,1}\left(J/4| \begin{array}{c} - \\ 0 \end{array}\right) = \frac{1}{4}e^{-J/4}.$$

The Mandel parameter $Q = 3$. Since $Q > 0$ for all $J \geq 0$ the photon distribution is super-Poissonian.

• *Pseudoharmonic oscillator* : An anharmonic potential suitable for the treatment of molecular vibrations is the pseudoharmonic oscillator (PHO)

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + V_p(r).$$

The effective potential of the PHO is

$$V_p(r) = \frac{m\omega^2}{8} r_0^2 \left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 + \frac{\hbar}{2m} p(p+1)\frac{1}{r^2}$$

where m is a reduced mass, ω angular frequency, r_0 the equilibrium distance between the nucli of the diatomic molecule and p rotational quantum numbers. $V_p(r)$ can be rewritten as

$$V_p(r) = \frac{m\omega^2}{8} r_p^2 \left(\frac{r}{r_p} - \frac{r_p}{r}\right)^2 + \frac{m\omega^2}{4}(r_p^2 - r_0^2)$$

where r_p is the changed equilibrium distance and it is given by

$$r_p = \left[\frac{2\hbar}{m\omega}(\beta^2 - \frac{1}{4})\right]^{\frac{1}{2}} \quad \text{where} \quad \beta = \left[(p + \frac{1}{2}) + \frac{m\omega r_0^2}{2\hbar}\right]^{\frac{1}{2}}.$$

The radial eigenfunctions and eigenvalues are given by

$$U_n^\beta = \frac{1}{B}\left[\frac{B^3 n!}{2^\beta \Gamma(n+\beta+1)}\right]^{\frac{1}{2}} (Br)^{\beta+\frac{1}{2}} e^{-B^2r^2/4} L_n^\beta(B^2r^2/2),$$

where L_n^β is the generalized Laguerre polynomial, and

$$E_{np} = \hbar\omega(n + \frac{1}{2}) + \frac{\hbar\omega\beta}{2} - \frac{m\omega^2 r_0^2}{4}$$

where $B = \sqrt{m\omega/\hbar}$. For $\beta = 2q - 1$ the eigenfunctions satisfy $\langle n, q \mid n', q\rangle = \delta_{nn'}$ and $\sum_{m=0}^{\infty} \mid n, q\rangle\langle n, q \mid= I$. For details see [11]. The spectrum is obtained from (3.1) with $k = 1$, $a_1 = \hbar\omega$ and $a_0 = \frac{\hbar\omega\beta}{2} - \frac{m\omega^2 r_0^2}{4}$. In (3.2) we set $\phi_n =\mid n, q\rangle$ and obtain $\rho(n) = \hbar^n\omega^n\Gamma(n+1)$, $\mathcal{N}(J)^2 = {}_0F_0(-;-;J/(\hbar\omega)) = e^{J/(\hbar\omega)}$ and

$$\lambda(J) = \frac{1}{4} G_{0,1}^{1,0}\left(J/(\hbar\omega)\middle| \begin{matrix} - \\ 0 \end{matrix}\right) = \frac{1}{\hbar\omega} e^{-J/(\hbar\omega)}.$$

The Mandel parameter $Q = \hbar\omega - 1$. If we rescale $\hbar$ and ω such that $\hbar\omega = 1$ we obtain the results of the Harmonic oscillator.

• *Infinite well* : In [2] GKCS were constructed for the infinite well potential

$$H = -\frac{\hbar}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2ma^2}.$$

The eigenfunctions form an orthonormal basis of the Hilbert space $\mathfrak{H} = L^2([0, \pi a], dx)$ and the eigenvalues are $E_n = \frac{\hbar^2}{2ma^2} n(n+2) = \hbar\omega n(n+2)$ where $\omega = \frac{\hbar}{2ma^2}$. In [2] GKCS were considered with $e_n = n(n+2)$; $n = 0, 1, \ldots$. To be compatible with (3.1) here we take $e_n = E_n$. This case can be viewed as a special case of (3.2) with $k = 2, a_2 = \hbar\omega, a_1 = 2\hbar\omega$, $a_0 = 0$ and $\phi_m =$ eigenfunction of the infinite well. Here we have

$\rho(n) = \hbar^n\omega^n n!(3)_n$, $\mathcal{N}(J)^2 = {}_0F_1(-;3;\frac{J}{\hbar\omega}) = 2\hbar\omega I_2(2\sqrt{J/(\hbar\omega)})/J$, where $I_\nu(x)$ is the modified Bessel function of the second kind,

$$\lambda(J) = \frac{1}{2\hbar\omega} G^{2,0}_{0,2}\left(J/(\hbar\omega)| \begin{array}{cc} & - \\ 2 & 0 \end{array}\right)$$

and

$$Q = \frac{2}{y^3}\left[\frac{(J+\hbar\omega)I_1(2y) + y(J-\hbar\omega)I_0(2y)}{{}_0F_1(-;3;y^2)}\right] - J - 1,$$

where $y = \sqrt{J/(\hbar\omega)}$. For $\hbar = \omega = 1$ we get

$$Q = \frac{2\left[(J+1)I_1(2\sqrt{J}) - \sqrt{J}I_0(2\sqrt{J})\right]}{\sqrt{J}I_0(2\sqrt{J}) - I_1(2\sqrt{J})}.$$

When $\hbar = \omega = 1$ one can numerically see that $Q < 0$ for very small values of J and it is positive for large values of J.

•*Pöschl-Teller* : In [2] GKCS were also constructed for the Pöschl-Teller potential

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{V_0}{2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2a} + \frac{\kappa(\kappa-1)}{\sin^2 x/2a}\right) - \frac{\hbar^2}{8ma^2}(\lambda+\kappa)^2, \quad 0 \le x \le \pi a$$

where$V_0 = \hbar^2/(4ma^2)$. The nondegenerate spectrum of this Hamiltonian is given by $E_n = \frac{\hbar^2}{2ma^2}n(n+\lambda+\kappa) = \hbar\omega e_n;\ n = 0,1,2,....$ The eigenfunctions form an orthonormal basis of the Hilbert space $L^2([0,\pi a], dx)$. In [2] GKCS were constructed for this Hamiltonian with $e_n = n(n+\lambda+\kappa);\ \lambda,\kappa > 1$. From (3.2), for the spectrum E_n, GKCS can be obtained with $k = 2, a_2 = \hbar\omega, a_1 = \hbar\omega(\lambda+\kappa), a_0 = 0$ and $\phi_n =$ eigenfunction of the Pöschl-Teller potential. In this case we have $\rho(n) = \hbar^n\omega^n n!(1+\lambda+\kappa)_n$,

$$\mathcal{N}(J)^2 = {}_0F_1(-;1+\lambda+\kappa;J/\hbar\omega) = \frac{I_{\lambda+\kappa}(2\sqrt{J/\hbar\omega})\Gamma(1+\lambda+\kappa)}{(\sqrt{J/\hbar\omega})^{\lambda+\kappa}},$$

$$\lambda(J) = \frac{1}{\Gamma(1+\lambda+\kappa)} G^{2,0}_{0,2}\left(J/(\hbar\omega)| \begin{array}{cc} & - \\ \lambda+\kappa & 0 \end{array}\right)$$

and

$$Q = \frac{2J}{y}\frac{I_{\lambda+\kappa+1}(2y)}{I_{\lambda+\kappa}(2y)} + \hbar\omega(\lambda+\kappa+1) - 1,$$

where $y = \sqrt{J/\hbar\omega}$. When $\hbar = \omega = 1$ we get

$$Q = 2\sqrt{J}\frac{I_{\lambda+\kappa+1}(2\sqrt{J})}{I_{\lambda+\kappa}(2\sqrt{J})} + \lambda + \kappa.$$

• *Eckart potential* : For $0 \le \beta x \le \pi$ and $A > B$, the energy spectrum of the Eckart potential,

$$H = -\frac{d^2}{dx^2} - A^2 + (A^2 + B^2 - A\beta)cosec^2(\beta x) - B(2A-\beta)cot(\beta x)cosec(\beta x)$$

is gicen by $E_n = \beta n(\beta n + 2A);\ \ n = 0,1,2,...$[4],[8]. This can be compared to (3.1) with $k = 2$, $a_2 = \beta^2$, $a_1 = 2A\beta$ and $a_0 = 0$. Since $\rho(n) = \beta^{2n}n!(\frac{\beta+2A}{\beta})_n$ the rest of the details follows from the Pöschl-Teller case with appropriate substitutions.

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